Problem Set 6

This sixth problem set explores the regular languages and their properties. This will be your first foray into computability theory, and I hope you find it fun and exciting!

As always, please feel free to drop by office hours, ask on Piazza, or email the staff list if you have any questions. We'd be happy to help out.

Good luck, and have fun!

Due Friday, February 23rd at 2:30PM

Problem One: Constructing DFAs

For each of the following languages over the indicated alphabets, construct a DFA that accepts precisely the strings that are in the indicated language. Your DFA does not have to have the fewest number of states possible, though for your own edification we'd recommend trying to construct the smallest DFAs possible.

Please use our online tool to design, test, and submit your answers to this problem. Handwritten or typed solutions will not be accepted. To use the tool, visit the CS103 website and click the "DFA/NFA Editor" link under the "Resources" header. If you're planning on submitting this assignment in a pair, in your GradeScope submission, please let us know the SUNetID (e.g. htiek or cbl, but not 06001234) of the partner who submitted the DFAs so that we can match the problem set to the submitted answers.

Unlike the programming assignments, you will not be able to see the results of the autograder when you submit. As a result, ***be sure to test your solutions thoroughly before you submit!***

i. For the alphabet $\Sigma = \{\texttt{a}, \texttt{b}, \texttt{c}\}$, construct a DFA for the language $\{ w \in \Sigma^* \mid w$ contains exactly two cs. $\}$

ii. For the alphabet $\Sigma = \{\texttt{a}, \texttt{b}\}$, construct a DFA for the language $\{ w \in \Sigma^* \mid w$ contains the same number of instances of the substring ab and the substring ba $\}$. Note that substrings are allowed to overlap, so aba $\in L$ and babab $\in L$.

iii. For the alphabet $\Sigma = \{\texttt{a}, \texttt{b}, \texttt{c}, \ldots, \texttt{z}\}$, construct a DFA for the language $\{ w \in \Sigma^* \mid w$ contains the word "cocoa" as a substring $\}$. *

Test your automaton thoroughly. This one has some tricky edge cases.

iv. Suppose that you are taking a walk with your dog along a straight-line path. Your dog is on a leash that has length two, meaning that the distance between you and your dog can be at most two units. You and your dog start at the same position. Consider the alphabet $\Sigma = \{\texttt{y}, \texttt{d}\}$. A string in Σ^* can be thought of as a series of events in which either you or your dog moves forward one unit. For example, the string "yydd" means that you take two steps forward, then your dog takes two steps forward. Let $L = \{ w \in \Sigma^* \mid w$ describes a series of steps that ensures that you and your dog are never more than two units apart $\}$. Construct a DFA for L.

Problem Two: Constructing NFAs

For each of the following languages over the indicated alphabets, construct an NFA that accepts precisely the strings that are in the indicated language. ***Please use our online system to design, test, and submit your automata***; see above for details. As before, ***please test your submissions thoroughly!***

i. For the alphabet $\Sigma = \{\texttt{a}, \texttt{b}, \texttt{c}\}$, construct an NFA for $\{ w \in \Sigma^* \mid w$ ends in a, bb, or ccc $\}$.

ii. For the alphabet $\Sigma = \{\texttt{a}, \texttt{b}, \texttt{c}, \texttt{d}, \texttt{e}\}$, construct an NFA for the language $\{ w \in \Sigma^* \mid$ the last character of w appears nowhere else in w, and $|w| \geq 1 \}$.

Stuck? Try reducing the alphabet to two or three letters and see if you can solve that version.

iii. For the alphabet $\Sigma = \{\texttt{a}, \texttt{b}\}$, construct an NFA for the language $\{ w \in \Sigma^* \mid w$ contains at least two b's with exactly five characters between them $\}$. For example, baaaaab is in the language, as is aabaabaaabbb and abbbbbabaaaaaaab, but bbbbb is not, nor are bbbab or aaabab.

* DFAs are often used to search large blocks of text for specific substrings, and several string searching algorithms are built on top of specially-constructed DFAs. The *Knuth-Morris-Pratt* and *Aho-Corasick* algorithms use slightly modified DFAs to find substrings extremely efficiently.

Problem Three: $\wp(\Sigma^*)$

Let Σ be an alphabet. Give a short English description of the set $\wp(\Sigma^*)$. Briefly justify your answer.

We think that there is a single "best answer." You should be able to describe the set in at most ten words.

Problem Four: Concatenation, Kleene Stars, and Complements

The regular languages are closed under a number of different operations. This problem explores some properties of those operations.

i. Prove or disprove: if L is a nonempty, finite language and k is a positive natural number, then $|L^k| = |L|^k$. Here, the notation $|L|^k$ represents "the cardinality of L, raised to the kth power," and the notation $|L^k|$ represents "the cardinality of the k-fold concatenation of L with itself."

Call back to formal definitions as you work through this problem. The most common mistakes we see people make on this problem involve making unfounded claims about set cardinalities. If you want to prove that two sets have the same cardinality, set up an explicit bijection between them.

ii. Prove or disprove: there is a language L where $\overline{(L^*)} = (\overline{L})^*$.

A good warm-up problem: what is $\emptyset^*$? Don't answer this question based off of your intuition; look back at the formal definition of the Kleene star.

Problem Five: Monoids and Kleene Stars

The Kleene star operator is one of the more unusual operators we've covered over the course of the quarter. This problem explores one of its fundamental properties.

Let Σ be an arbitrary alphabet. A ***monoid over Σ*** is a set $M \subseteq \Sigma^*$ with the following properties:

$$\varepsilon \in M \qquad\qquad \forall x \in M.\ \forall y \in M.\ xy \in M.$$

Let L be an arbitrary language over Σ and let M be an arbitrary monoid over Σ. Prove that if $L \subseteq M$, then $L^* \subseteq M$.

In the course of writing this proof, please call back to the formal definition of language concatenation and the Kleene star, and use induction as appropriate. Here's a refresher on the definitions:

$$L_1L_2 = \{\ wx \mid w \in L_1 \text{ and } x \in L_2\ \}$$

$$L^0 = \{\varepsilon\} \qquad\qquad L^{n+1} = LL^n$$

$$L^* = \{\ w \mid \exists n \in \mathbb{N}.\ w \in L^n\ \}$$

The result that you've shown here is a building block toward a larger result: the Kleene star of a language L is the smallest monoid containing all the strings in L.

This problem is all about finding the right way to formalize things. Think about, ultimately, what it is that you need to prove. Before you start trying to prove that, break the task down into smaller pieces and make sure you organize everything in a way that makes the logical flow easy to read and rigorously covers all cases. Once you have the setup put together, dive in and fill out each section.

Problem Six: Hard Reset Sequences

A ***hard reset sequence*** for a DFA is a string w with the following property: starting from any state in the DFA, if you read w, you end up in the DFA's start state.

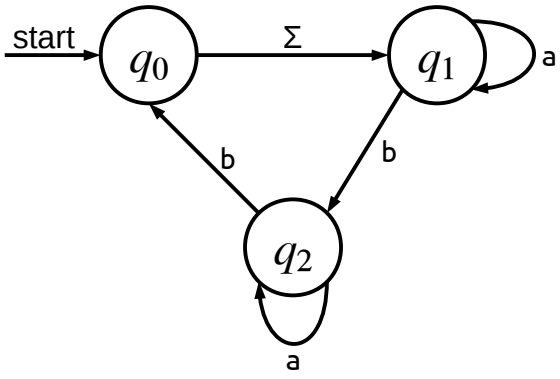
Hard reset sequences have many practical applications. For example, suppose you're remotely controlling a Mars rover whose state you're modeling as a DFA. Imagine there's a hardware glitch that puts the Mars rover into a valid but unknown state. Since you can't physically go to Mars to pick up the rover and fix it, the only way to change the rover's state would be to issue it new commands. To recover from this mishap, you could send the rover a hard reset sequence. Regardless of what state the rover got into, this procedure would guarantee that it would end up in its initial configuration.

Here is an algorithm that, given any DFA, will let you find every hard reset sequence for that DFA:

1. Add a new start state q_s to the automaton with ε-transitions to every state in the DFA.
2. Perform the subset construction on the resulting NFA to produce a new DFA called the ***power automaton***.
3. If the power automaton contains a state corresponding solely to the original DFA's start state, make that state the only accepting state in the power automaton. Otherwise, make every state in the power automaton a rejecting state.

This process produces a new automaton that accepts all the hard reset sequences of the original DFA. It's possible that a DFA won't have any hard reset sequences (for example, if it contains a dead state), in which case the new DFA won't accept anything.

Apply the above algorithm to the following DFA and give us a hard reset sequence for that DFA. For simplicity, please give the subset-constructed DFA as a transition table rather than a state-transition diagram. We've given you space for the table over to the right, and to be nice, we've given you exactly the number of rows you'll need.



	a	b

Sample hard reset sequence: _____

Finding a hard reset sequence for this DFA is a lot easier if you take a few minutes to think about what the power automaton does.

Problem Seven: Complementing NFAs

In lecture, we saw that if you take a DFA for a language L and flip all the accepting and rejecting states, you end up with a DFA for $\overline{L}$.

Draw a simple NFA for a language L where flipping all the accepting and rejecting states does not produce an NFA for $\overline{L}$. Briefly justify your answer; you should need at most a sentence or two.

Problem Eight: DFAs, Formally

When we first talked about graphs, we saw them first as pictures (objects connected by lines), but then formally defined a graph G as an ordered pair (V, E), where V is a set of nodes and E is a set of edges. This rigorous definition tells us what a graph actually is in a mathematical sense, rather than just what it looks like.

We've been talking about DFAs for a while now and seen how to draw them both as a collection of states with transitions (that is, as a state-transition diagram) and as a table with rows for states and columns for characters. But what exactly *is* a DFA, in a mathematical sense?

Formally speaking, a DFA is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- Q is a finite set, the elements of which we call *states*;
- Σ is a finite, nonempty set, the elements of which we call *characters*;
- $\delta : Q \times \Sigma \to Q$ is the ***transition function***, described below;
- $q_0 \in Q$ is the start state;
- $F \subseteq Q$ is the set of accepting states.

The transition function warrants a bit of explanation. When we've drawn DFAs, we've represented the transitions either by arrows labeled with characters or as a table with rows and columns corresponding to states and symbols, respectively. In this formal definition, the transition function δ is what ultimately specifies the transition. Specifically, for any state $q \in Q$ and any symbol $a \in \Sigma$, the transition from state q on symbol a is given by $\delta(q, a)$.

This question explores some properties of this rigorous definition.

i. Is it possible for a DFA to have no states? If so, define a DFA with no states as a 5-tuple, explaining why your 5-tuple meets the above requirements. If not, explain why not.

ii. Is it possible for a DFA to have no *accepting* states? If so, define a DFA with no accepting states as a 5-tuple, explaining why your 5-tuple meets the above requirements. If not, explain why not.

iii. In class, we said that a DFA must obey the rule that for any state and any symbol, there has to be exactly one transition defined on that symbol. What part of the above definition guarantees this?

iv. Is it possible for a DFA to have an unreachable state (that is, a state that is never entered regardless of what string you run the DFA on)? If so, define a DFA with an unreachable state as a 5-tuple, explaining why your 5-tuple meets the above requirements. If not, explain why not.

Defining a DFA requires you to define a transition function δ. You should define that function the same way that we've defined all other functions this quarter: either give a rule, define it as a piecewise function, or draw a picture. If you've having trouble doing so, it might mean that you picked too complex of a DFA and might want to search for something simpler.

Problem Nine: Why the Extra State?

In our proof that the regular languages are closed under the Kleene closure operator (that is, if L is regular, then L^* is regular), we used the following construction:

1. Begin with an NFA N where $\mathscr{L}(N) = L$.
2. Add in a new start state q_{start}.
3. Add an ε-transition from q_{start} to the start state of N.
4. Add ε-transitions from each accepting state of N to q_{start}.
5. Make q_{start} an accepting state.
6. Make every state besides q_{start} a rejecting state.

You might have wondered why we needed to add q_{start} as a new state to the NFA. It might have seemed more natural to do the following:

1. Begin with an NFA N where $\mathscr{L}(N) = L$.
2. Add ε-transitions from each accepting state of N to the start state of N.
3. Make the start state of N an accepting state.
4. Make every other state of N a rejecting state.

Unfortunately, this doesn't work correctly. Find a language L and an NFA N for L such that using the second construction does not create an NFA for L^*. Justify why the language of the new NFA isn't L^*.

The best way to get a handle on this problem is to work through a bunch of small, concrete examples of applying both constructions. Make sure you can articulate why, intuitively, each construction seems like it's a plausible way to form an automaton for the Kleene star of the language of the original automaton. Once you can do that, see if you can identify something potentially problematic about the second approach. Then, based on your understanding of that flaw, play around and see if you can build an NFA where that flaw leads to the construction not working properly.

Optional Fun Problem 1: Why Finite? (1 Point Extra Credit)

A ***deterministic infinite automaton***, or ***DIA***, is a generalization of a DFA in which the automaton has infinitely many different states. Formally speaking, a DIA is given by the same 5-tuple definition as a DFA from Problem Eight, except that Q must be an infinite set. Since DIAs have infinitely many states, they're mostly an object of purely theoretical study.

Prove that if L is an arbitrary language over an alphabet Σ, then there is a DIA that accepts L (that is, the DIA accepts every string in L and rejects every string not in L.) To do so, show how to start with a language L, formally define a 5-tuple corresponding to a DIA for L, then formally prove that that DIA accepts all and only the strings in L.

Optional Fun Problem 2: Edit Distances (1 Point Extra Credit)

The ***edit distance*** between two strings w and x is the minimum number of edits that need to be made to w to convert it into x. Here, an ***edit*** consists of either adding a character somewhere into w, deleting a character somewhere from w, or replacing a character of w with another. For example, `cat` and `dog` have edit distance 3, `table` and `maple` have edit distance 2, and `edit` and `distance` have edit distance 6.

Let Σ = {`w`, `h`, `i`, `m`, `s`, `y`}. Design an NFA for { $w \in \Sigma^*$ | the edit distance of w and `whimsy` is at most three }. Submit your answer online, and let us know in your written assignment who made the submission.